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Existence of Many Ergodic Absolutely Continuous Invariant Measures for Piecewise-Expanding C^2 Chaotic Transformations in $\mathbb{R}^2$ on a Fixed Number of Partitions

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Let Ω be a region in $\mathbb{R}^n$ and let $\mathscr{P} = \{P_i\}_{i=1}^m$ be a partition of Ω into a finite number of closed subsets having piecewise C^2 boundaries of finite $(n-1)$-dimensional measure. Let $\tau: \Omega \to \Omega$ be piecewise C^2 on $\mathscr{P}$ where, $\tau_i = \tau_{|P_i}$ is a C^2 diffeomorphism onto its image, and expanding in the sense that there exists $\alpha > 1$ such that for any $i = 1, 2,..., m$ $\|D\tau_i^{-1}\| < \alpha^{-1}$, where $D\tau_i^{-1}$ is the derivative matrix of τ_i^{-1} and $\|\cdot\|$ is the Euclidean matrix norm. By means of an example, we will show that the simple bound of one-dimensional dynamics cannot be generalized to higher dimensions. In fact, we will construct a piecewise expanding C^2 transformation on a fixed partition with a finite number of elements in $\mathbb{R}^2$, but which has an arbitrarily large number of ergodic, absolutely continuous invariant measures.

KEY WORDS: Absolutely continuous invariant measures (acim); ergodic; piecewise-C^2; expanding transformation; perturbation.

1. INTRODUCTION

Let Ω be a bounded region in $\mathbb{R}^n$ and let $\mathscr{P} = \{P_i\}_{i=1}^m$ be a partition of Ω into a finite number of subsets having piecewise C^2 boundaries of finite $(n-1)$-dimensional measure. Let $\tau: \Omega \to \Omega$ be piecewise C^2 on $\mathscr{P}$ where, $\tau_i = \tau_{|P_i}$ is a C^2 diffeomorphism onto its image, and expanding in the sense that there exists $\alpha > 1$ such that for any $i = 1, 2,..., m$ $\|D\tau_i\| < \alpha^{-1}$, where $D\tau_i$ is the derivative matrix of τ_i and $\|\cdot\|$ is the euclidean matrix norm. Then, under general conditions [Adl], it can be shown that τ has an

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absolutely continuous invariant measure (**acim**). The result in [Adl] is a generalization of the results proved in [Jab], [Kel], [Can] and [G-B].

In this note, we investigate the problem of finding an upper bound on the number of **acim**'s for higher dimensional transformations.

For one-dimensional transformations [Li-Y], $\tau: I \to I$, $I = [0, 1]$, it is well known that the number of discontinuities of $\tau'(x)$ provides an upper bound for the number of independent **acim**. This result has been improved in [Boy], [Pia], [B-H] and [B-B]. The key to all these bounds lies in the fact that invariant densities for piecewise C^2 expanding transformations are of bounded variation. In one dimension, a density of bounded variation is bounded and it can be proved that its support consists of a finite union of closed intervals. A simple argument then shows that at least one point of discontinuity of τ' must lie in the largest closed interval, which will provide an upper bound on the number of **acim**. In higher dimensions, the much more complicated geometrical setting and the complex form of the definition of bounded variation [Giu] do not permit an easy generalization of the one-dimensional result. For example, in two dimensions, the variation in one direction is integrated along the other direction. It is this integration which allows a function of bounded variation in $\mathbb{R}^n$ to be unbounded and its support to be devoid of interior.

In general, dynamical systems can have a large set of invariant measures. For example, higher dimensional point transformation models for cellular automata [G-B2], can have many **acim**.

In 1990, Góra, Boyarsky and Proppe [G-B-P], outlined the possibility of constructing a piecewise expanding C^2 transformation on a fixed partition in $\mathbb{R}^2$ with a finite number of elements which has an arbitrarily large number of ergodic **acim**. There the use of certain triangles having a particular geometry as the supports of an ergodic **acim** is suggested. By means of a sketch, it is outlined (without proof, however) that it is possible to take care of the trapezoidal regions between triangles satisfying all conditions. Although the conjecture turns out to be correct, we will see that the construction cannot be done in a simple manner.

We use the triangles suggested in [G-B-P] as supports of ergodic **acim**. For the trapezoidal regions between the triangles, we use another set of triangles which are not supports of **acim**, satisfying the following conditions:

(1) The triangles are mapped in an expanding manner similar to that of the triangles which are supports of ergodic **acim**, and

(2) the intersection of images of triangles which are supports of ergodic **acim** and images of triangles which are not supports of ergodic **acim** is empty.

This reduces the trapezoidal regions to rectangular regions. We will then, by the aid of Lemma 1, map each such rectangular region to a "tube" in a C^2 and expanding manner in such a way that the tube does not intersect the images of the triangles that support the ergodic **acim**.

Finally, by making small perturbations to these maps near the "vertical" edges of these rectangular regions, we can obtain a map that is C^2 and expanding on all of S_1 (respectively S_{-1}) (see Fig. 2).

MAIN RESULTS

We will construct a piecewise expanding C^2 transformation on a fixed partition with 10 elements which has an arbitrarily large number of ergodic **acim**.

Lemma 1. For $L>0$ large enough, there exists an expanding C^2 diffeomorphism of a rectangle R with sides L and 1 into a tube $\mathscr{T}$ similar to the one shown in Fig. 1.

Remark. Although this may seem obvious, the point of this note is to provide a complete proof of the main result.

Proof. Let P be a C^2 function defined on an interval $[x_0, L+x_0]$ satisfying

$$P(x_0)=0, \quad P(L+x_0)=-1, \quad P'(x_0)=1, \quad P'(L+x_0)=1 \tag{1}$$

The graph of P is a curve of length αL, for some $\alpha>1$,which we also denote by P. We parametrize P by arc length $s=\alpha x$:

$$x \longmapsto (u(\alpha x), v(\alpha x)),$$

where

$$v(s)=P\circ u(s) \tag{2}$$

and

$$u'^2(s)+v'^2(s)=1 \tag{3}$$

We shift the curve P along the u-axis: $(u(\alpha x), v(\alpha x)) \longmapsto ((1+u(\alpha x), v(\alpha x))$. The mapping of the rectangle R to the tube $\mathscr{T}$ is constructed by moving along the curve P a distance of αx and then along its upward normal vector a distance of $\sqrt{2}\,y$ for each point $(x, y)\in R$:

Let $\vec{n}$ be the upward normal vector to the curve P, i.e,

$$\vec{n}(s)=(n_1(s), n_2(s))=(-v'(s), u'(s)) \tag{4}$$

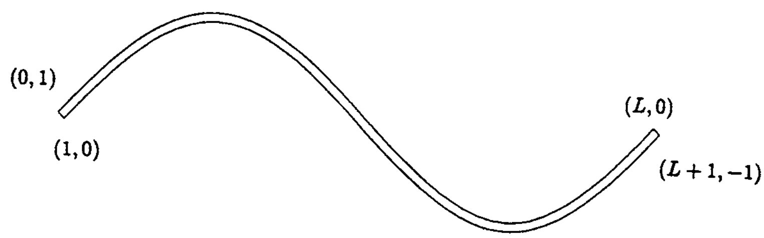

Figure 1

To derive an expression for $\vec{n}$ in terms of P, first differentiate (2) with respect to arc length s, which implies that

$$v'(s) = P' \circ u(s)\, u'(s) \tag{5}$$

Using (5) in (3), we get

$$u'(s) = \frac{1}{\sqrt{1 + (P' \circ u(s))^2}} \qquad v'(s) = \frac{P' \circ u(s)}{\sqrt{1 + (P' \circ u(s))^2}} \tag{6}$$

Now using (6) in (4), we get

$$\vec{n}(s) = (n_1(s), n_2(s)) = \left(\frac{-P' \circ u(s)}{\sqrt{1 + (P' \circ u(s))^2}}, \frac{1}{\sqrt{1 + (P' \circ u(s))^2}} \right) \tag{7}$$

Now note that the width of the tube in the direction of $\vec{n}$ is $\sqrt{2}$; the transformation maps vertical sections of $[x_0, L + x_0] \times [0, 1]$ linearly onto segments in the $\vec{n}$ direction. Therefore the two-dimensional transformation τ from the rectangle to the tube is given by

$$\tau(x, y) = (1 + u(\alpha x) + \sqrt{2}\, y n_1(s), v(\alpha x) + \sqrt{2}\, y n_2(s)).$$

Thus, the Jacobian matrix of τ is

$$J_\tau = \begin{pmatrix} \alpha u'(s) + \sqrt{2}\, y \dfrac{\partial n_1}{\partial x} & \sqrt{2}\, n_1(s) \\ \alpha v'(s) + \sqrt{2}\, y \dfrac{\partial n_2}{\partial x} & \sqrt{2}\, n_2(s) \end{pmatrix} \tag{8}$$

Calculating $\partial n_1/\partial x$ and $\partial n_2/\partial x$ in terms of P and its derivatives, using (7), we obtain

$$\frac{\partial n_1}{\partial x}=\frac{-\alpha P''\circ u\cdot u'}{(1+(P'\circ u)^2)^2}\,\frac{\partial n_2}{\partial x}=\frac{-\alpha P''\circ u\cdot P'\circ u\cdot u'}{(1+(P'\circ u)^2)^2} \tag{9}$$

where the independent variable $s=\alpha x$ is suppressed for notational simplicity.

The unit tangent vector $\vec{T}$ at the point (u, v) of the curve P is: $\vec{T}=(u', v')=(u', P'\circ u\cdot u')$.

Recall that the curvature κ (which we define to be non-negative) is given by

$$\kappa^2=\left|\frac{d\vec{T}}{ds}\right|^2$$

where

$$\frac{d\vec{T}}{ds}=(u'', P'\circ u\cdot u''+P''\circ u\cdot(u')^2).$$

Thus, we have

$$\kappa^2=\left|\frac{d\vec{T}}{ds}\right|^2=u''^2+(P'\circ u\cdot u''+P''\circ u(u')^2)^2 \tag{10}$$

Using (6), we get

$$u''=\frac{-P''\circ uP'\circ u\cdot u'}{(1+(P'\circ u)^2)^{3/2}}\qquad\text{and}\qquad\kappa^2=\frac{(P''\circ u)^2}{(1+(P'\circ u)^2)^3} \tag{11}$$

Thus

$$\kappa(s)=\frac{|P''\circ u(s)|}{(1+(P'\circ u(s))^2)^{3/2}} \tag{12}$$

We express $\partial n_1/\partial x$ and $\partial n_2/\partial x$ in terms of κ in the case $P''>0$ (respectively $P''<0$):

$$\frac{\partial n_1}{\partial x}=\mp\alpha\kappa\cdot n_2\qquad\frac{\partial n_2}{\partial x}=\pm\alpha\kappa\cdot n_1 \tag{13}$$

Using (13) in (8), the Jacobian matrix of τ becomes

$$J_\tau = \begin{pmatrix} \alpha(u'(s) \mp \sqrt{2}\, y\kappa n_2) & \sqrt{2}\, n_1 \\ \alpha(v'(s) \pm \sqrt{2}\, y\kappa n_1) & \sqrt{2}\, n_2 \end{pmatrix}.$$

Using (4) and (5) it follows that

$$J_\tau = u'(s) \begin{pmatrix} \beta & -\sqrt{2}\, P' \circ u(s) \\ \beta P' \circ u(s) & \sqrt{2} \end{pmatrix}$$

where $\beta = \alpha(1 \mp \sqrt{2}\, y\kappa)$ for $P'' > 0$ (respectively $P'' < 0$). Therefore, the eigenvalues of the Jacobian matrix will be

$$\lambda = \frac{1}{2\sqrt{1 + (P' \circ u(s))^2}} \left((\beta + \sqrt{2}) \pm \sqrt{(\beta - \sqrt{2})^2 - 4\sqrt{2}\, \beta (P' \circ u(s))^2}\right).$$

Now we note that if $(\beta - \sqrt{2})^2 < 4\sqrt{2}\, \beta(P' \circ u(s))^2$, then the eigenvalues are complex and

$$|\lambda_1| = |\lambda_2| = \frac{1}{2\sqrt{1 + (P' \circ u(s))^2}} \times \sqrt{(\beta + \sqrt{2})^2 + 4\sqrt{2}\, \beta(P' \circ u(s))^2 - (\beta - \sqrt{2})^2},$$

and if $(\beta - \sqrt{2})^2 \geqslant 4\sqrt{2}\, \beta(P' \circ u(s))^2$, then the eigenvalues are real and

$$|\lambda_{\min}| = \frac{1}{2\sqrt{1 + (P' \circ u(s))^2}} \left((\beta + \sqrt{2}) - \sqrt{(\beta - \sqrt{2})^2 - 4\sqrt{2}\, \beta(P' \circ u(s))^2}\right).$$

For $1.001 < \beta < 1.45$ and $0 \leqslant |P' \circ u(s)| \leqslant 1.05$ both eigenvalues are strictly greater than 1 in absolute value, i.e. the mapping of the rectangle to the tube is expanding. □

Let R^{-1} (respectively $\mathscr{T}^{-1}$) denote the reflection of R (respectively $\mathscr{T}$) about the x-axis. Let $\mathscr{T}^v$ = reflection of $\mathscr{T}$ about the vertical line $x = L/2 + x_0$ and $\mathscr{T}^{-v} = (\mathscr{T}^v)^{-1} = (\mathscr{T}^{-1})^v$.

Note that the construction of $\mathscr{T}^{-1}$ from R^{-1} is isometric to the construction of $\mathscr{T}$ from R.

We now give an example of a function P which satisfies the criteria of Lemma 1, and which will be used in a later construction.

Example 1. Let

$$P(t)=\frac{L}{2\pi}\sin\frac{2\pi}{L}(t-1)+\frac{1}{2}\cos\frac{\pi}{L}(t-1)-\frac{1}{2}$$

where L is to be determined later, and $1 \leqslant t \leqslant L+1$. Then:

$$P'(t)=\cos\frac{2\pi}{L}(t-1)-\frac{\pi}{2L}\sin\frac{\pi}{L}(t-1)$$

We note that $P(1)=0$ $P(L+1)=-1$ and $P'(1)=P'(L+1)=1$ (i.e., Eq. (1) is satisfied).

Also note that,

$$P''(t)=\frac{-2\pi}{L}\sin\frac{2\pi}{L}(t-1)-\frac{\pi^2}{2L^2}\cos\frac{\pi}{L}(t-1)$$

We have the following estimates for κ:

$$|P'(t)|<1+\frac{\pi}{2L}, \qquad \kappa \leqslant |P''(t)| \leqslant \frac{2\pi}{L}+\frac{\pi^2}{2L^2}=\frac{4\pi L+\pi^2}{2L^2}$$

The amount of expansion applied to the line to produce the curve P is given by:

$$\alpha=\frac{1}{L}\int_1^{L+1}\sqrt{1+P'^2(t)}\,dt$$

It is easy to see that for $L \geqslant 100$, $\beta \in (1.1, 1.33)$ and $|P'|<1.039$; therefore, if we choose $L \geqslant 100$ then both eigenvalues will be larger than 1 in absolute value.

By $\triangle pqr$ we denote the triangle with vertices at the points p, q and r, and by $\square pqrs$ we denote the rectangle with vertices at the points p, q, r and s.

We are now ready to establish the main result: the construction of arbitrarily many ergodic **acim** for a piecewise expanding C^2 transformation in $\mathbb{R}^2$ on a fixed partitition. The idea in [G-B-P] of using triangles as ergodic sets of **acim** is essential in this construction.

Theorem 1. For any number k there exists a two dimensional piecewise C^2 expanding transformation with 10 elements which has at least k ergodic **acim**.

Proof. We prove the theorem by the means of a construction. Consider the following 10 elements partition of Ω: where $\ell(k)=2k+1+2kL$

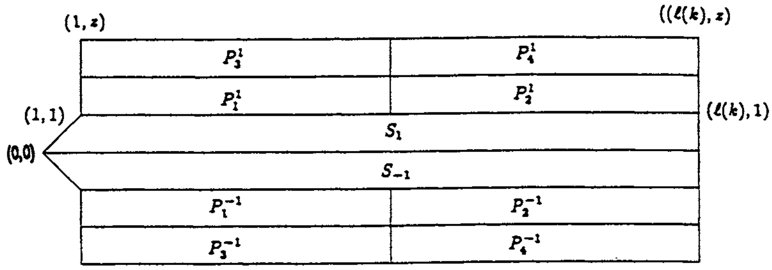

Figure 2

and L is to be determined later, where z is large enough so that for every $(x, y) \in \mathscr{T}$ defined in Lemma 1 we have $y \leqslant z$.

Let $P^1 = \square(1, 1)(\ell(k), 1)((\ell(k), z)(1, z)$ and P^{-1} be its reflection about the x-axis. Each P^j is subdivided into 4 rectangles P_i^j for $1 \leqslant i \leqslant 4$ and $j = -1, 1$ as shown in Fig. 2. The exact manner of subdivision is irrelevant. Let $\tau(P_i^j) = P^j$, $1 \leqslant i \leqslant 4$ where τ maps each rectangle P_i^j linearly onto the large rectangle P^j.

Thus, it remains to define τ on S_1 and S_{-1} (see Fig. 2 for definition of S_1 and S_{-1}). Now we define the sets E_i for $1 \leqslant i \leqslant 2k+1$ on $S_1 \cup S_{-1}$ which are the sets that will produce supports for the $k+1$ ergodic **acim**. Let $E_i = \triangle a_i^0 b_i^1 b_i^{-1}$ where, $a_i = (i-1)L + i - 1$, $a_i^j = (a_i, j)$ and $b_i^j = (a_i + 1, j)$ for $j \in \{-1, 0, 1\}$ and $1 \leqslant i \leqslant 2k+1$, we also define $c_i^0 = (a_i + 2, 0)$ $1 \leqslant i \leqslant 2k$, see Fig. 4.

We define the triangles which are the supports of ergodic **acim** as follows:

$$T_i^j = E_i \cap S_j \qquad 1 \leqslant i \leqslant 2k+1, \qquad j \in \{1, -1\}$$

see Fig. 3.

Let $R_i = \square a_i^{-1} a_i^1 b_i^1 b_i^{-1}$. We define the triangles which are not supports of ergodic **acim** as follows:

$$\tilde{T}_i^j = (R_i \cap S_j) \backslash E_i \qquad 2 \leqslant i \leqslant 2k+1, \qquad j \in \{-1, 1\}$$

see Fig. 3.

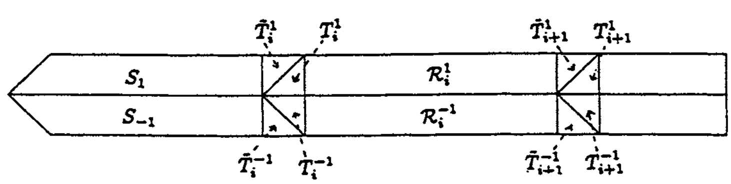

Figure 3

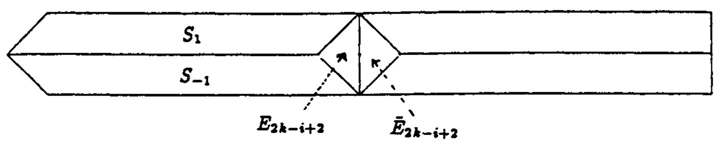

Figure 4

For $1 \leqslant i \leqslant 2k$ let $\tilde{E}_i = \triangle b_i^{-1} b_i^1 c_i^0$, see Fig. 4. Let

$$v_j = \begin{pmatrix} \ell(k) \\ j(-iL + L - i) \end{pmatrix}, \qquad M_1 = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad M_{-1} = \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$$

Then for a point $p \in R_i \cap S_j$ we define

$$\tau(p) = v_j + M_j \cdot p$$

Then

$$\tau(T_i^j) = E_{2k-i+2} \qquad \text{and} \qquad \tau(\tilde{T}_i^j) = \tilde{E}_{2k-i+2}$$

and

$$\tau(b_i^0) = a_{2k-i+2}^0 \qquad \tau(a_i^0) = b_{2k-i+2}^{-j} \qquad \tau(b_i^j) = b_{2k-i+2}^j \qquad \tau(a_i^j) = c_{2k-i+2}^0$$

see Figs. 3 and 4.

Note that in this construction τ is continuous across the boundaries between the triangles, since τ is an affine map on $R_i \cap S_j = T_i^j \cup \tilde{T}_i^j$.

Remark 1. In what will follow $\mathscr{T}_i^j$ and $\mathscr{R}_i^j$ correspond to $\mathscr{T}$ and R in Lemma 1.

Let $\mathscr{R}_i^j = \square b_i^0 a_{i+1}^0 a_{i+1}^j b_i^j$, and define $\tau(\mathscr{R}_i^j) = \mathscr{T}_{2k-i+2}^{jv}$ (see Lemma 1 and Fig. 5).

We note that the condition $P'(x_0) = P'(L + x_0) = 1$, together with the construction of $\tau(x, y)$ in Lemma 1, is used to glue the pieces of τ together in a continuous manner, since the construction was defined so that for each rectangle, the map τ of Lemma 1 and the affine maps defined on the adjacent triangles match on the common vertical sides. This guarantees that τ is C^0 on all of S_1 (respectively S_{-1}). We also note that τ is C^2 and expanding on all of S_1 (respectively S_{-1}), except on the common boundaries of triangles and rectangles, for the following reasons:

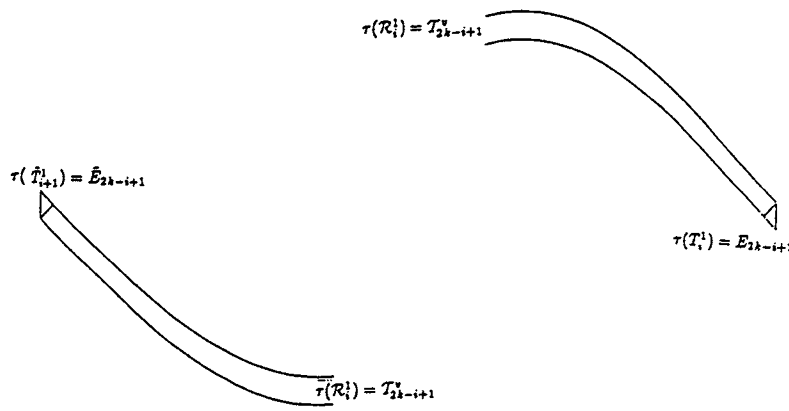

Figure 5

(1) $\tau_{|R_i \cap S_j}$, $\tau_{|\mathscr{R}_i^j}$ and $\tau_{|R_{i+1} \cap S_j}$ are C^2 on the closure of their domains ($R_i \cap S_j$, $\mathscr{R}_i^j$ and $R_{i+1} \cap S_j$ respectively) and coincide on $R_i \cap S_j \cap \mathscr{R}_i^j$ and $R_{i+1} \cap S_j \cap \mathscr{R}_i^j$.

(2) $\tau_{|R_i \cap S_j}$ and $\tau_{|R_{i+1} \cap S_j}$ are affine maps with expansion constant equal to $\sqrt{2}$, and $\tau_{|\mathscr{R}_i^j}$ is expanding by Lemma 1 and Example 1 for $L > 100$.

Now for each $1 \leqslant i \leqslant 2k+1$ and $j = -1, 1$ we perturb τ slightly in $\mathscr{R}_i^j$ near its vertical edges so that the resulting map is C^2 and expanding on all of S_j. We use Theorem 2.5 in Chapter 2 of [Hir], after a slight modification of τ. Given $\mathscr{R}_i^j$ and the triangles T_i^j (resp. $\tilde{T}_{i+1}^j$) adjacent to it on the left (resp right), and given $\varepsilon > 0$, we first extend the maps on the triangles to a small portion of each end of $\mathscr{R}_i^j$ as follows. Let $\mathscr{R}_i^j = [0, L] \times [0, 1]$ (in local coordinates); we redefine τ on $((0, \varepsilon) \times [0, 1]) \cup ((L - \varepsilon, L) \times [0, 1])$ to be simply the (affine) maps on the respective triangles. The reason for this extension is that we do not want to perturb the mapping on the triangles; the extension ensures that the points where τ fails to be C^2 are a positive distance from the boundaries of the triangles. We now redefine the map τ of Lemma 1 so that the domain is $[\varepsilon, L - \varepsilon] \times [0, 1]$ instead of $[0, L] \times [0, 1]$ (still in local coordinates). We note that doing this does not change the validity of Lemma 1, although the map τ is slighly perturbed. It is clear from the construction in Lemma 1 that the resulting map (also denoted by τ) on the domain $D_i^j = \mathscr{R}_i^j \cup T_i^j \cup \tilde{T}_{i+1}^j$ is continuous, but in general fails to be C^2 on the line segments $\{\varepsilon\} \times [0, 1]$ and $\{L - \varepsilon\} \times [0, 1]$. To apply the theorem, we work with the smaller rectangle $A_i^j = [\frac{1}{4}\varepsilon, L - \frac{1}{4}\varepsilon] \times [0, 1]$. Let $U = (0, L) \times (-\frac{1}{4}\varepsilon, 1 + \frac{1}{4}\varepsilon)$. Then U is open and $\Omega \supset U \supset A_i^j$. We extend τ to U so that the extended map (which we still denote by τ) is C^2 except possibly on a subset of $\{\varepsilon, L-\varepsilon\} \times (-\frac{1}{4}\varepsilon, 1+\frac{1}{4}\varepsilon)$. This can be done because the map τ of Lemma 1 could have been defined in precisely the same way on a slightly thicker rectangle. The actual extension is immaterial; once we have applied the theorem, we restrict the perturbed map to the domain D_i^j and restore the original definition of τ on the portion of U outside of this domain. Now let $K=([\frac{1}{2}\varepsilon, \frac{3}{2}\varepsilon] \cup [L-\frac{3}{2}\varepsilon, L-\frac{1}{2}\varepsilon]) \times [0,1]$. Let $W=((\frac{3}{4}\varepsilon, \frac{5}{4}\varepsilon) \cup (L-\frac{5}{4}\varepsilon, L-\frac{3}{4}\varepsilon))(-\frac{1}{4}\varepsilon, 1+\frac{1}{4}\varepsilon)$. We note that τ is C^2 on a neighbourhood of $K-W$. The conditions of the theorem are thus satisfied, so that for any neighbourhood $\mathcal{N} \subset C_S^0(U, \mathring{\Omega})$ of τ there is a map $\tilde{\tau}$ which is C^2 on a neighbourhood of K and which equals τ on $U-W$(here $C_S^0(U, \mathring{\Omega})$ denotes the space of continuous functions from U to $\mathring{\Omega}$, the interior of Ω, with the strong topopology). Since τ itself is already C^2 on $U-W$, these two properties imply that $\tilde{\tau}$ is C^2 on all of A_i^j (in fact on U, except possibly on a subset of $(U-A_i^j) \cap W$). Thus the restriction of $\tilde{\tau}$ to A_i^j, together with the undisturbed map on the remaining portion of the domain D_i^j, is C^2 on the entire domain. For a suitably small ε and for $\tilde{\tau}$ sufficiently close to τ, it is clear that the C^2 map $\tilde{\tau}|_{D_i^j}$ is also expanding on all of D_i^j.

To complete the proof let

$$\mathscr{E}_i = E_i \cup E_{2k-i+2} \qquad \text{for} \quad 1 \leqslant i \leqslant k+1$$

Then by the above construction, we have $\tau(\mathscr{E}_i) = \mathscr{E}_i = \tau^{-1}(\mathscr{E}_i)$, for $1 \leqslant i \leqslant k+1$, so each $\mathscr{E}_i$ is an invariant set of positive Lebesgue measure. On each $\mathscr{E}_i$, τ is piecewise expanding and onto. By [Man, Chapter III, Theorem 1.3], each $\mathscr{E}_i$ supports exactly one ergodic **acim**. Since there are $k+1$ distinct $\mathscr{E}_i$, the proof is complete. □

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